THE ERDÖS-SZÜSZ-TURÁN DISTRIBUTION FOR EQUIVARIANT PROCESSES

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ABSTRACT. We resolve problems posed by Kesten and Erdös-Szüsz-Turán on probabilistic Diophantine approximation via methods of homogeneous dynamics. Our methods allows us to generalize the problem to the setting of general measure-valued processes in $\mathbb{R}^n$, and obtain applications to the distribution of point sets which occur in higher-dimensional Diophantine approximation and the geometry of translation surfaces.

CONTENTS

1. INTRODUCTION

1.1. **Dirichlet's Theorem.** The most classical result in Diophantine approximation is Dirichlet's Theorem, which is stated in two forms: First, for all irrational $\alpha \in [0, 1]$, and any integer $Q \geq 1$, there is a $1 \leq q \leq Q$ and a $p \in \mathbb{Z}$ relatively prime to q so that

$$\text{(D1)} \qquad |\alpha q - p| \leq \frac{1}{Q}.$$

As a corollary, one obtains that there are infinitely many $\frac{p}{q} \in \mathbb{Q}$ (with $gcd(p, q) = 1$) satisfying

$$\text{(D2)} \qquad \left| \alpha - \frac{p}{q} \right| \leq \frac{1}{q^2}.$$

In this paper, we consider the *distribution* of the number of solutions to modified versions of (D1) and (D2) from a probabilistic perspective initiated by Erdös-Szüsz-Turán [14] and Kesten [24]. Our methods show how this problem can be generalized and solved in many other geometric and number theoretic settings.

1.2. **The Erdös-Szüsz-Turán and Kesten distributions.** In 1958 [14], Erdös-Szüsz-Turán introduced a problem in probabilistic Diophantine approximation: what is the probability $f(N, A, c)$ that a point α chosen from the uniform distribution on $[0, 1]$ has a solution $\frac{p}{q} \in \mathbb{Q}$ to the modified Dirichlet equation

$$(1.1) \qquad \left| \alpha - \frac{p}{q} \right| \leq \frac{A}{q^2},$$

with denominator $q \in [N, cN]$? Here $A > 0, c > 1$ are fixed positive parameters, and N is a parameter which goes to infinity. We note that by a well known result of Hurwitz, $A = \frac{1}{\sqrt{5}}$ is the best allowable constant so that (1.1) has infinitely many solutions for all α. Let $\mathrm{EST}(A, c, N)$ denote the random variable counting the number of solutions to (1.1). Then, the Erdös-Szüsz-Turán question is the existence of the limit

$$\lim_{N \to \infty} P(\mathrm{EST}(A, c, N) > 0).$$

Considering analagously a modified version of (D1), Kesten [24] defined the sequence of random variables $K(A, N)$ (our notation differs from [24]) as the number of solutions to

$$(1.2) \qquad |\alpha q - p| \leq \frac{A}{N}, 1 \leq q \leq N.$$

We consider the following

Question. *What (if they exist) are the limiting distributions of the random variables* $\mathrm{EST}(A, c, N)$ *and* $K(A, N)$*?*

Our main results shows that the limiting distribution exists and can be viewed as the probability of a random unimodular lattice intersecting a certain fixed region. Let $X_2 = \mathrm{SL}(2, \mathbb{R}) / \mathrm{SL}(2, \mathbb{Z})$ denote the space of unimodular lattices in $\mathbb{R}^2$, via the identification

$$g \, \mathrm{SL}(2, \mathbb{Z}) \mapsto g\mathbb{Z}^2.$$

Let μ_2 denote the Haar probability measure on X_2, and given $\Lambda \in X_2, \Lambda = g\mathbb{Z}^2$, let Λ_{prim} be the set of primitive vectors in Λ.

Theorem 1.1. *The limiting distribution of the random variables* $\mathrm{EST}(A, c, N)$ *and* $K(A, N)$ *exist and denoting the random variables with these limiting distributions as* $\mathrm{EST}(A, c)$ *and* $K(A)$*, we have*

$$(1.3) \qquad P(\mathrm{EST}(A, c) = k) = \mu_2(\Lambda \in X_2 : \#(\Lambda_{\mathrm{prim}} \cap H_{A,c}) = k),$$

and

$$(1.4) \qquad P(K(A) = k) = \mu_2(\Lambda \in X_2 : \#(\Lambda_{\mathrm{prim}} \cap R_A) = k)$$

where

$$(1.5) \qquad H_{A,c} = \{(x, y) \in \mathbb{R}^2 \ : \ xy \leq A, 1 \leq y \leq c\},$$

and

$$(1.6) \qquad R_A = \{(x, y) \in \mathbb{R}^2 \ : \ |x| \leq A, 0 \leq y \leq 1\}.$$

Expressing the limiting distributions as distributions on the space of lattices allows us to apply classical results on the geometry of numbers to obtain moment and concentration estimates (Theorem 2.4). Furthermore, our translation of the problem into a geometric and dynamical problem is axiomatic and flexible, and as we will see in §2 applies to point sets associated to linear forms and translation surfaces. Our proof also shows that the same result holds as long as α is chosen from a probability measure with a continuous density (and in higher dimensions, certain natural classes of singular measures). Our main result (Theorem 4.1) describes how to define, construct and compute related distributions in the general setting of equivariant processes, which we define precisely in §4.

1.3. **History.** This circle of problems has a long history, starting with the original paper of Erdös-Szüsz-Turán [14]. There, in addition to posing the problem of the limiting probability of $P(\mathrm{EST}(A, c, N) > 0)$ (they denoted this putative limit as $f(A, c)$), they showed that for $A \le \frac{c}{1+c^2}$, that the limit existed and

$$f(A, c) = \frac{12}{\pi^2} A \log c.$$

Subsequently Kesten [24] showed that the the limit exists under the assumption $Ac \le 1$, and this assumption was removed by Kesten-Sos [25]. Explicit formulas for $f(A, c)$ were computed much more recently independently by Boca [10] and Xiong-Zaharescu [48], who also considered localizing α to smaller intervals. Our results offer a significantly more refined picture of $\mathrm{EST}(A, c)$ and $K(A)$, yielding the limiting distribution (not just the probability of positivity) of both random variables in both this setting and in a variety of other geometric and number theoretic contexts, and our methods can localize α to even *shrinking* intervals (§1.5.2). Recently, related results have been studied from the point of view of deviation of ergodic averages for toral translations, see Dolgopyat-Fayad [17, 18] for intriguing distributional results from this perspective. Our perspective is inspired by the work of Marklof-Strömbergsson [31, 32], who gave several beautiful applications of homogeneous dynamics to understanding the fine scale statistics of point sets.

1.4. **Organization.** In the remainder of this introduction, we prove Theorem 1.1, using equidistribution of horocycles on the space X_2. In §2, we describe our general results in the settings of lattices; linear forms; diophantine approximation on curves, and translation surfaces. In §4 we describe our general philosophy and state our main axiomatic theorem. In §5, we prove our results in their various incarnations on the space of lattices, and in §6, we prove our results in the setting of translation surfaces. In §7, we remark on how our approach can be applied in even more situations, including (but not limited to) multiplicative diophantine approximation; complex diophantine approximation; the distribution of cut-and-project quasicrystals; and the distribution of discrete lattice orbits on the Clifford plane and related problems on cusp excursions on hyperbolic manifolds.

1.5. **Equidistribution on the modular surface.** We relate the Erdös-Szüsz-Turán and Kesten distributions to dynamics on the space of unimodular lattices, and prove Theorem 1.1. This approach will generalize to higher dimensions, and will allow us derive a number of results using appropriate equidistribution results. We begin with the original question of Erdös-Szüsz-Turán and Kesten, proving Theorem 1.1.

1.5.1. *Proof of Theorem 1.1.* We note $\text{EST}(A, c, N) = k$ if and only if

$$\exists \text{ exactly } k \text{ distinct } \frac{p}{q} \in \mathbb{Q} \text{ such that } N \leq q \leq cN \text{ and } \left| \alpha - \frac{p}{q} \right| < \frac{A}{q^2}.$$

Equivalently, there are exactly k vectors

$$\begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{Z}^2_{\text{prim}} \text{ such that } \begin{pmatrix} x \\ y \end{pmatrix} := u_\alpha \begin{pmatrix} p \\ q \end{pmatrix} \in g_{\log N} H_{A,c},$$

where

$$u_\alpha = \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix} \text{ and } g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

This follows by rewriting

$$\left| \alpha - \frac{p}{q} \right| < \frac{c}{q^2}, N \leq q \leq cN$$

as

$$q|q\alpha - p| < A,$$

and then as

$$|xy| < A, N \leq y \leq cN.$$

Thus, we are interested in the measure of the set of $\alpha \in [0, 1]$ satisfying

(1.7) $$\#(g_{\log N} u_\alpha \mathbb{Z}^2_{\text{prim}} \cap H_{A,c}) = k.$$

Let χ_k denote the indicator function of the set

$$\{\Lambda \in X \ : \ \#(\Lambda_{\text{prim}} \cap H_{A,c}) = k\}.$$

To compute $P(\text{EST}(A, c) = k)$, we are interested in the $N \to \infty$ behavior of

(1.8) $$\int_0^1 \chi_k(g_{\log N} u_\alpha \mathbb{Z}^2_{\text{prim}}) d\alpha.$$

Let η_N denote the measure $d\alpha$ on the set $\{g_{\log N} u_\alpha \mathbb{Z}^2_{\text{prim}} : 0 \leq \alpha \leq 1\}$, so we can rewrite (1.8) as $\eta_N(\chi_K)$. By Zagier's equidistribution theorem [49, p. 279], we have

$$\eta_N \longrightarrow \mu_2,$$

as $N \to \infty$, where the convergence is in the weak-$*$ topology. The functions χ_k can be approximated by continuous functions with compact support on the space of lattices X_2, so by a standard approximation argument we conclude

$$P(\text{EST}(A, c, N) = k) = \eta_N(\chi_K) \underset{N \to \infty}{\longrightarrow} \mu_2(\chi_K).$$

For the Kesten distribution, we note by a similar argument that $K(A, N) = k$ if and only if

$$\#\left(g_{\log N} u_\alpha \mathbb{Z}^2_{\text{prim}} \cap R_A\right) = k.$$

Thus, proceeding as above, we obtain (1.4).

$\square$

1.5.2. *Measures and windows.* The proof of Theorem 1.1 in fact yields much more information. A strengthening of Zagier's theorem due to Shah [37] allows us to obtain the equdistribution result for any absolutely continuous measure on $[0, 1]$. Thus, the limiting random variables EST and K do not depend on the initial distribution of α (as long as it is continuous). A different strengthening of Zagier's result is due to Hejhal [22] (with subsequent work of Strombergsson [42]), which implies that we can sample α from smaller subintervals depending on N, as long as the subintervals shrink no faster than $N^{-1/2}$.

We can also consider the Erdös-Szüsz-Turán and Kesten distributions associated to solutions of (1.1) and (1.2) with $c_1 N \le q \le c_2 N$, with $0 < c_1 < c_2$. The limiting distributions will again be given by the probability random lattices intersect fixed regions in k points, with the regions being given by

$$H_{A,c_1,c_2} = \{(x,y) \in \mathbb{R}^2 \ : \ xy \le A, c_1 \le y \le c_2\}$$

and

$$R_{A,c_1,c_2} = \{(x,y) \in \mathbb{R}^2 \ : \ |x| \le A, c_1 \le y \le c_2\}.$$

1.5.3. *Moments and concentration.* To compute moments of the Erdös-Szüsz-Turán and Kesten distribution, we need to understand the quantities

$$\sum_{k=0}^{\infty} k^t P(X = k), t \in \mathbb{R},$$

where X is either $\mathrm{EST}(A, c)$ or $K(A)$. For $t = 1$, we can rewrite this as

$$\int_{X_2} \# \left(\Lambda_{\mathrm{prim}} \cap H_{A,c} \right) d\mu_2(\Lambda).$$

By the Siegel mean value theorem [40], we have

$$\int_{X_2} \# \left\{ \Lambda_{\mathrm{prim}} \cap H_{A,c} \right\} d\mu_2(\Lambda) = \frac{6}{\pi^2} |H_{A,c}| = \frac{12}{\pi^2} A \log C.$$

This is the *expected* number of solutions to (1.7) (in the $N \to \infty$ limit). Note that if $A \le \frac{c}{1+c^2}$,

$$P(\mathrm{EST}(A, c) > 0) = \sum_{k \ge 1} P(\mathrm{EST}(A, c) = k) = \frac{12}{\pi^2} A \log C,$$

so we have

$$E(\mathrm{EST}(A, c)) = \sum_{k \ge 1} k P(\mathrm{EST}(A, c) = k) = \sum_{k \ge 1} P(\mathrm{EST}(A, c) = k),$$

so for $k > 1$, $P(\mathrm{EST}(A, c) = k) = 0$, that is, the Erdös-Szüsz-Turán distribution is concentrated at 1. For the Kesten distribution, similar computations yield the mean,

$$E(K(A)) = \frac{6}{\pi^2} |R_A| = \frac{6A}{\pi^2}.$$

In the setting of higher-dimensional Diophantine approximation, we will obtain bounds on higher moments via classical results on the geometry of numbers.

2. Erdős-Szüsz-Turán and Kesten distributions in higher dimensions

2.1. **Diophantine approximation.** We start with a natural higher dimensional generalization of the original Erdős-Szüsz-Turán problem. Let $d \geq 2 = m + 1$ ($d = 2, m = 1$ corresponds to our original problem), and fix $A > 0, c > 1$ and a norm $\| \cdot \|$ on $\mathbb{R}^m$. Let $\mathbf{x}$ be chosen from the uniform distribution on $[0, 1]^m$, and let $\text{EST}_d(A, c, N)$ denote the number of solutions $(\mathbf{p}, q) \in \mathbb{Z}^m \times \mathbb{Z}$ (with $(\mathbf{p}, q)$ primitive) to the modified Dirichlet equation

$$(2.1) \qquad \|\mathbf{x}q - \mathbf{p}\| \leq A q^{-\frac{1}{d}},$$

with $q \in [N, cN]$, and $K_d(A, N)$ denote the number of solutions to

$$(2.2) \qquad \|\mathbf{x}q - \mathbf{p}\| \leq A N^{-\frac{1}{d}},$$

with $q \in [1, N]$.

Theorem 2.1. *The limiting distributions of* $\text{EST}_d(A, c, N)$ *and* $K_d(A, N)$ *exist and the distributions of the limiting random variables* $\text{EST}_d(A, c)$ *and* $K_d(A)$ *are given by*

$$P(\text{EST}_d(A, c)) = \mu_d(\Lambda \in X_d : \#(\Lambda_{\text{prim}} \cap H_{d, A, c}) = k)$$

and

$$P(K_d(A)) = \mu_d(\Lambda \in X_d : \#(\Lambda_{\text{prim}} \cap R_{d, A}) = k)$$

where

$$(2.3) \qquad H_{d, A, c} = \{(\mathbf{x}, y) \in \mathbb{R}^d \times \mathbb{R} \ : \ \|\mathbf{x}\| y \leq A, 1 \leq y \leq c\}$$

and

$$(2.4) \qquad R_{d, A} = \{(\mathbf{x}, y) \in \mathbb{R}^d \times \mathbb{R} \ : \ \|\mathbf{x}\| \leq A, 0 \leq y \leq 1\}$$

2.2. **Linear Forms.** Next, we consider systems of linear forms. Let $d = m + n$, $m, n \geq 1$, fix $A > 0, c > 1$, and norms $\| \cdot \|_m$ and $\| \cdot \|_n$ on $\mathbb{R}^m$ and $\mathbb{R}^n$. We consider the set of m linear forms in n variables, parameterized by $M_{m \times n}(\mathbb{R})$, the set of $m \times n$ real matrices. We identify $M_{m \times n}(\mathbb{R})$ with $\mathbb{R}^{mn}$. Let X be chosen from the uniform distribution on $[0, 1]^{mn}$. We define the random variable $\text{EST}_{m \times n}(A, c, N)$ as the number of solutions $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times \mathbb{Z}^n$ (with $(\mathbf{p}, \mathbf{q})$ primitive) to the modified Dirichlet equation

$$(2.5) \qquad \|X\mathbf{q} - \mathbf{p}\|_m \leq A \|\mathbf{q}\|_n^{-\frac{n}{m}},$$

with $\|q\|_n \in [N, cN]$. Similarly, we define $K_{m \times n}(A, N)$ as the number of solutions to

$$(2.6) \qquad \|X\mathbf{q} - \mathbf{p}\|_m \leq A |N|^{-\frac{n}{m}},$$

with $\|q\|_n \in [1, N]$.

Theorem 2.2. *The limiting distributions of the random variables* $\text{EST}_{m \times n}(A, c, N)$ *and* $K_{m \times n}(A, N)$ *exist and the distributions of the limiting random variables* $\text{EST}_{m \times n}(A, c)$ *and* $K_{m \times n}(A)$ *are given by*

$$P(\text{EST}_{m \times n}(A, c) = k) = \mu_d(\Lambda \in X_d : \#(\Lambda_{\text{prim}} \cap H_{m \times n, A, c}) = k)$$

and

$$P(K_{m \times n}(A) = k) = \mu_d(\Lambda \in X_d : \#(\Lambda_{\text{prim}} \cap R_{m \times n, A}) = k)$$

$$(2.7) \qquad H_{m \times n, A, c} = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m \times \mathbb{R}^n \ : \ \|\mathbf{x}\|_m \|\mathbf{y}\|_n \leq A, 1 \leq \|\mathbf{y}\|_n \leq c\}.$$

$$(2.8) \qquad R_{m \times n, A} = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m \times \mathbb{R}^n \ : \ \|\mathbf{x}\|_m \leq A, 0 \leq \|\mathbf{y}\|_n \leq 1\}.$$

We note that Theorem 2.1 is a special case of the above Theorem with $n = 1$.

2.3. **Approximation on Curves.** The subject of metric Diophantine approximation *on manifolds* studies typical Diophantine properties of points on manifolds. It is well known and easy to see using the Borel Cantelli Lemma, that almost every real number is *not very well approximable.* This means that the inequality

$$|qx - p| < 1/|q|^{1+\epsilon}$$

has at most finitely many solutions. This result generalises easily to arbitrary dimension. In 1932, K. Mahler conjectured that almost every point on the curve $(x, x^2, \ldots, x^n)$ is not very well approximable. Mahler's conjecture started the subject and there have been many subsequent works, including recent dramatic advances due to Kleinbock-Margulis, Beresnevich, Velani, and others. The constraint of lying on a manifold makes the subject considerably more complicated than classical Diophantine approximation. Nevertheless, our approach can be used to compute Erdös-Szüsz-Turán and Kesten distributions for vectors lying on curves. Let $d = n + 1$, and $\phi : [a, b] \to \mathbb{R}^n$ be an analytic curve whose image is not contained in a proper affine subspace, and $\| \cdot \|$ denote a norm on $\mathbb{R}^{n-1}$. Let x be chosen from the uniform distribution on $[a, b]$, and let $\mathrm{EST}_\phi(A, c)$ denote the random variable counting solutions to

$$\|q\phi(x) - \mathbf{p}\| < Aq^{-\frac{1}{d}}, N \leq q \leq cN.$$

Let $K_\phi(A, N)$ denote the random variable counting solutions to

$$\|q\phi(x) - \mathbf{p}\| < AN^{-\frac{1}{d}}, 1 \leq q \leq N.$$

Theorem 2.3. *The random variables* $\mathrm{EST}_\phi(A, c, N)$ *and* $K_\phi(A, N)$ *have limiting distributions, and the limiting random variables* $\mathrm{EST}_\phi(A, c)$ *and* $K_\phi(A)$ *have distributions given by*

$$P(\mathrm{EST}_\phi(A, c) = k) = \ \mu \left\{ \Lambda \in X_d \ : \ \#(\Lambda_{\mathrm{prim}} \cap H_{d, A, c}) = k \right\},$$

and

$$P(K_\phi(A, c) = k) = \ \mu \left\{ \Lambda \in X_d \ : \ \#(\Lambda_{\mathrm{prim}} \cap R_{d, A}) = k \right\},$$

where $H_{d, A, c}$ *and* $R_{d, A}$ *are as in Theorem 2.1*

Remark: There is also an analogue of Theorem 2.2 for curves in the space of linear $\mathbb{R}^{mn}$, which is exactly parallel to Theorem 2.3.

2.4. **Measures and windows.** As in the setting of 1-dimensional approximation, we can also work with q (or $\|q\|$) in appropriate subranges of the form $[c_1 N, c_2 N]$ with appropriate changes to the limiting distributions (replacing the y range with $[c_1, c_2]$). Additionally, choosing absolutely continuous measures also does not change the limiting distribution.

2.5. **Moments and concentration.** Classical results from the geometry of numbers allow us to compute moments of the random variables EST and K. We recall the definition of the *Siegel transform*: given $f \in C_c(\mathbb{R}^d)$ and $\Lambda \in X_d$ define

$$\widehat{f}(\Lambda) = \sum_{v \in \Lambda_{\mathrm{prim}}} f(v).$$

Siegel showed

$$\int_{X_d} \widehat{f} d\mu_d = \frac{1}{\zeta(d)} \int_{\mathbb{R}^d} f dm,$$

where f is Lebesgue measure. Thus, the expectation of the random variables EST and K is given by $\frac{1}{\zeta(n)}$ times the volume of the regions $H_d A, c$ and $R_d(A)$. Building on Siegel's work, Rogers [35] and Schmidt [41] computed bounds for higher moments of $\widehat{f}$. These can be exploited to give precise moment estimates for EST and K, and yield non-trivial concentration phenomenon. For the variance, their results yield:

Theorem 2.4. *(Rogers* [35, Theorem 4]*, Schmidt* [41, Theorem 3]*, see also* [8, Lemma 4.3]*) Let X be either $\mathrm{EST}_{m \times n}(A, c)$ or $K_{m \times n}(A)$, and let $\mu = E(X)$. There is a constant C_d, depending only on dimension $d = m + n$, so that*

$$V(X) = E((X - \mu)^2) \le C_d \mu.$$

In particular, for any $T > 0$,

$$P(|X - \mu| > T\sqrt{\mu}) \le \frac{C_d}{T^2}.$$

Proof. The second assertion is an immediate consequence of the first. For the first, we note both $\mathrm{EST}_{m \times n}$ and $K_{m \times n}$ are random variables counting the number of lattice points in a bounded set ($H_{m \times n}(A, c)$ and $R_{m \times n}(A)$). By [8, Lemma 4.3] (which is essentially contained in Rogers), we have that for any random variables of this type,

$$E(X^2) \le \mu^2 + C_d \mu.$$

In fact C_d can be chosen to be $8\zeta(d-1)/\zeta(d)$ (for $d \ge 3$).

□

Remarks: It would be interesting to use Rogers' [35] results on higher moments (bounds on L^p norm for $p < d$) to obtain further concentration results. Note that since $R_{m \times n}(A)$ grows polynomially in A and $H_{m \times n}(A, c)$ polynomially in c and logarithmically in A, so do the expectations of EST and K, which gives explicit concentration bounds in terms of A and c. Finally, we note that Kesten [24, Theorem 3] considered the $d \to \infty$ limit of K_d and proved Poisson behavior (under appropriate normalizations) using the method of moments.

3. TRANSLATION SURFACES

Our approach also yields information on the geometry of the set of holonomy vectors of saddle connections on translation surfaces. Given $g \ge 1$, an *translation surface S of genus g* is a pair $S = (X, \omega)$, where X is a compact Riemann surface of genus g and ω is a holomorphic 1-form. A *saddle connection γ* on S is a geodesic

(in the flat metric determined by ω) connecting two zeros of ω, with none in its interior. The holonomy vector of γ is defined by

$$z_\gamma := \int_\gamma \omega \in \mathbb{C}.$$

The set

$$\Lambda_S := \{z_\gamma : \gamma \text{ a saddle connection on } S\}$$

is a discrete subset of $\mathbb{R}^2$ with quadratic growth (cf. Masur [33]), that is there are constants $0 < c_1 \leq c_2$ so that

$$c_1 R^2 \leq \#(\Lambda_S \cap B(0, R)) \leq c_2 R^2.$$

We define the moduli space Ω_g of translation surfaces by considering equivalence classes of translation surfaces up to biholomorphism. This space is decomposed into strata $\mathcal{H}(\alpha)$ consisting of holomorphic differentials with zeros of order $\alpha_1, \ldots, \alpha_k$, where $\alpha = (\alpha_1, \ldots, \alpha_k)$ is an integer partition of $2g - 2$. Each stratum consists of at most 3 connected components [29], and there is a natural Lebesgue probability measure $\mu_{\mathcal{H}}$ on the each stratum $\mathcal{H}(\alpha)$, known as Masur-Veech measure.

There is a natural $\mathrm{SL}(2, \mathbb{R})$-action on the space Ω_g which respects the decomposition into strata, and acts ergodically on each connected component of a stratum [34, 43]. The set Λ_S varies equivariantly under this action, that is

$$\Lambda_{gS} = g\Lambda_S,$$

where $\mathrm{SL}(2, \mathbb{R})$ acts on $\mathbb{R}^2$ by the usual linear action. The fine-scale geometry of the sets Λ_S has been a subject of much recent investigation [2, 3, 46, 47], and our approach allows us to define Erdös-Szüsz-Turán and Kesten distributions associated to translation surfaces. We note that for $g = 1$, $\Omega_1 = X_2$, so this setting is another natural generalization of the original Erdös-Szüsz-Turán and Kesten problems.

Let $\theta \in [0, 2\pi)$ be chosen from the uniform distribution. We want to understand how well vectors in Λ_S approximate the direction θ, in terms of their length. Given $A > 0, c > 1, N > 0$, define the random variables $\mathrm{EST}(S, N)$ and $K(S, N)$ by

$$\mathrm{EST}(S, N) = \#\left(r_\theta \Lambda_S \cap H_{A,c,N}\right)$$

and

$$K(S, N) = \#\left(r_\theta \Lambda_S \cap R_{A,N}\right),$$

where

$$H_{A,c,N} = \{(x, y) \in \mathbb{R}^2 \ : \ xy \leq A, N \leq y \leq cN\}$$

and

$$R_{A,N} = \{(x, y) \in \mathbb{R}^2 \ : \ |x| \leq A, 0 \leq y \leq N\}.$$

We say $S_0 \in \Omega_g$ has *circle limit measure* μ on Ω_g if the measures $d\theta$ on $\{g_t r_\theta S\}_{0 \leq \theta < 2\pi}$ converge to μ. A result of Nevo [15] shows that for any stratum $\mathcal{H}$, $\mu_{\mathcal{H}}$-a.e. $S \in \mathcal{H}$ has circle limit measure $\mu_{\mathcal{H}}$.

Theorem 3.1. *Suppose $S_0 \in \Omega_g$ has circle limit measure μ_0. Then $\mathrm{EST}(S_0, N)$ and $K(S_0, N)$ both have limiting distributions, and denoting the random variables with this limiting distribution by $\mathrm{EST}(S)$ and $K(S)$, we have*

$$P(\mathrm{EST}(S_0) = k) = \mu_0\left(S \in \Omega_g : \#\left(\Lambda_S \cap H_{A,c} = k\right)\right)$$

and

$$P(K(S_0) = k) = \mu_0\left(S \in \Omega_g : \#\left(\Lambda_S \cap R_A = k\right)\right).$$

In particular, for any stratum $\mathcal{H}$ and $\mu_{\mathcal{H}}$-a.e. $S_0 \in \mathcal{H}$,

$$P(\mathrm{EST}(S_0) = k) = \mu_{\mathcal{H}}\left(S \in \mathcal{H} : \#\left(\Lambda_S \cap H_{A,c}\right) = k\right)$$

and

$$P(K(S_0) = k) = \mu_{\mathcal{H}}\left(S \in \mathcal{H} : \#\left(\Lambda_S \cap R_A\right) = k\right).$$

3.1. **Lattice Surfaces.** For particular highly symmetric surfaces, we can say more. We denote the stabilizer of the point $S = (X, \omega) \in \Omega_g$ under the $\mathrm{SL}(2, \mathbb{R})$ action by $\mathrm{SL}(X, \omega)$. A translation surface S is called a *lattice surface* (also known as an *Veech surface*) if $\mathrm{SL}(X, \omega)$ is a lattice. The lattices that occur are always nonuniform, and the $\mathrm{SL}(2, \mathbb{R})$ orbit of S is closed, a copy of $\mathrm{SL}(2, \mathbb{R})/\mathrm{SL}(X, \omega)$ in Ω_g. For these surfaces, we have

Theorem 3.2. *Suppose $S_0 = (X_0, \omega_0)$ is a lattice surface, and write $\Gamma = \mathrm{SL}(X_0, \omega_0)$. Let μ_{Γ} denote the Haar probability measure on $\mathrm{SL}(2, \mathbb{R})/\Gamma$. Then $\mathrm{EST}(S_0, N)$ and $K(S_0, N)$ both have limiting distributions, and denoting the random variables with this limiting distribution by $\mathrm{EST}(S_0)$ and $K(S_0)$, we have*

$$P(\mathrm{EST}(S_0) = k) = \mu_{\Gamma}\left(g\Gamma \in \mathrm{SL}(2, \mathbb{R})/\Gamma : \#\left(g\Lambda_{S_0} \cap H_{A,c} = k\right)\right)$$

and

$$P(K(S_0) = k) = \mu_{\Gamma}\left(g\Gamma \in \mathrm{SL}(2, \mathbb{R})/\Gamma : \#\left(g\Lambda_{S_0} \cap R_A = k\right)\right).$$

3.2. **Expectation.** To compute the expectation of the random variables K and EST in this setting, we use the *Siegel-Veech* formula [44]. This states that for any $\mathrm{SL}(2, \mathbb{R})$-invariant measure μ on $\mathcal{H}$ where the *Siegel-Veech transform*

$$\widehat{f}(S) = \sum_{z \in \Lambda_S} f(z)$$

is in $L^1(\mu)$ for any $f \in C_c(\mathbb{R}^2)$, there is a constant (the *Siegel-Veech constant*) c_{μ} so that

$$\int_{\mathcal{H}} \widehat{f}(S) = c_{\mu} \int_{\mathbb{R}^2} f \, dm,$$

where m is Lebesgue measure on $\mathbb{R}^2$. Applying this to our situation, we say that the expectations of our limiting random variables is given by a scalar multiple of the area of the sets $H_{A,c}$ and R_A, depending on the circle limit measure. The computation of Siegel-Veech constants is an active and challenging area of research, see, for example [16] for seminal work. In the setting of lattice surfaces, Veech [44] related these constants to the covolume of $\mathrm{SL}(X, \omega)$ in $\mathrm{SL}(2, \mathbb{R})$.

4. Equivariant Processes

In this section we define the axiomatic setup of equivariant measure-valued processes and state our main result Theorem 4.1. This perspective is inspired by the work of W. Veech [44] and J. Marklof, as well as that of A. Eskin and H. Masur [15]. It has the great advantage that once we make the proper definitions, the proof of the main theorems are essentially tautologies. The power of the method lies in its flexibility: we will see that the axioms can be verified in several different situations.

4.1. **Equivariant measure processes.** Let $n \geq 2$, and $G \subset GL(d, \mathbb{R})$. Let (X, μ) denote a Borel-G-space together with a G-invariant Borel probability measure μ. A $(G\text{-})$*equivariant measure process* (also known as a *Siegel measure*, see [44]) is a triple (X, μ, ν) where ν is a map

$$\nu : X \to \mathcal{M}(\mathbb{R}^d)$$

from X to the space $\mathcal{M}(\mathbb{R}^d)$ of σ-finite Radon Borel measures on $\mathbb{R}^d$ satisfying the equivariance condition

$$\nu(gx) = g_* \nu(x)$$

for all $g \in G, x \in X$, where G acts linearly on $\mathbb{R}^d$.

4.2. **Erdös-Szüsz-Turán distributions.** Given a sequence of equivariant measure processes $\mathbb{X} = \{(X_N, \eta_N, \nu_N)\}$ and a Borel subset $\mathcal{R} \subset \mathbb{R}^d$ we define the *Erdös-Szüsz-Turán distribution* $\eta = \eta(\mathbb{X}, \mathcal{R})$ on $\mathbb{R}^+$ as the measure given by (if the the limit exists)

$$\eta(\mathbb{X}, \mathcal{R})(0, t) = \lim_{N \to \infty} \eta_N(x \in X : \nu_N(x)(\mathcal{R}) \leq t).$$

4.3. **Equidistribution.** Our main result concerns the setting where our sequence $\mathbb{X} = \{X, \eta_N, \nu\}$, that is, a sequence of measures η_n on a fixed G-space X together with an assignment ν.

Theorem 4.1. *Suppose $\eta_N \to \mu$ (in the weak-* topology). Then*

$$\eta(\mathbb{X}, \mathcal{R})(0, t) = \mu(x \in X : \nu(x)(\mathcal{R}) \leq t).$$

Proof. By assumption, our measures are all Radon Borel measures, so if $\eta_N \to \mu$ in the weak-* topology, a standard approximation arguments gives that for any (fixed) Borel measurable subset $B \subset X$,

$$\eta_N(B) \longrightarrow \mu(B).$$

Applying this to $B = \{x \in X : \nu(x)(\mathcal{R}) \leq t\}$, we have our result. $\square$

This theorem, as stated, is a tautology. The key to applying it is finding appropriate equidistribution results that allow one to take a natural sequence $\mathbb{X}$ and find a limiting measure so that $\eta_N \to \mu$.

4.4. **Orbits and point processes.** In many of our applications, the measures η_N will be supported on orbits of subgroups $H \subset G$ and be the push-forward of some measure on H under the orbit map. In addition, ν will often be a *point process*, that is, the assignment of a discrete set with counting measure.

5. Equidistribution on the space of lattices

We prove our main Diophantine results using equidistribution results for flows on the space of unimodular lattices. Let μ_d denote the Haar probability measure on

$$X_d = \mathrm{SL}(d, \mathbb{R}) / \mathrm{SL}(d, \mathbb{Z}).$$

X_d is the space of unimodular (covolume 1) lattices in $\mathbb{R}^d$, via the identification

$$g \, \mathrm{SL}(d, \mathbb{Z}) \longmapsto g \mathbb{Z}^d.$$

Given $\Lambda = g\mathbb{Z}^d \in X$, we say that $v \in \Lambda$ is *primitive* if

$$v = gw, w \in \mathbb{Z}^d \backslash \{0\}, \gcd(w) = 1$$

and denote by Λ_{prim} the set of primitive points in Λ. For all of our Diophantine results, we will use the equivariant assignment

$$g\,\mathrm{SL}(d,\mathbb{Z}) \longmapsto \sum_{v \in g\mathbb{Z}^d_{\text{prim}}} \delta_v,$$

which, in the notation of §4 we view as a map $\nu_d : X_d \to \mathcal{M}(\mathbb{R}^d)$. Let m, n be positive integers and let $d = m + n$. Set

$$G = \mathrm{SL}(d,\mathbb{R}), \Gamma = \mathrm{SL}(d,\mathbb{Z}), u_X = \begin{pmatrix} \mathrm{Id}_m & X \\ 0 & \mathrm{Id}_n \end{pmatrix}, H = \{u_X \; : \; X \in \mathrm{Mat}_{m \times n}(\mathbb{R})\}.$$

The group H is the expanding horospherical subgroup of G with respect to

(5.1) $g_t = \mathrm{diag}(e^{t/m}, \ldots, e^{t/m}, e^{t/n}, \ldots, e^{t/n}), t > 0.$

The following Lemma is a straightforward generalisation of the the argument in the introduction and allows us to interpret the Erdös-Szüsz-Turán and Kesten distributions in terms of homogeneous dynamics.

Lemma 5.1. *Let notation be as above. Then*

$$\mathrm{EST}(A, c, N) = k \text{ if and only if } \# \left(g_{\log N} u_X \mathbb{Z}^d_{\text{prim}} \cap H_{A,c} \right) = k.$$

We can therefore proceed as before. Let η_N denote the measure dY on the set $\{g_{\log N} u_Y \mathbb{Z}^d_{\text{prim}} : 0 \leq \|Y\| \leq 1\}$. It is well known that

$$\eta_N \longrightarrow \mu_d,$$

as $N \to \infty$, where the convergence, as before, is in the weak-$*$ topology. This seems to date to Rogers [35, p.250, (4)], who claims the result (without proof) (see Rogers [36, Chapter 4], for a proof of an averaged version). We refer the reader to Kleinbock-Margulis [26] where a stronger statement, with a rate of convergence is proved. We note that, Zagier's theorem, used in the introduction also comes with a rate, however the rate of convergence in these equidistribution statements does not shed additional light on the Erdös-Szüsz-Turán distribution. Let χ_k denote the indicator function of the set

$$\{\Lambda \in X \; : \; \# \left(\Lambda_{\text{prim}} \cap H_{A,c} = k \right)\}.$$

The functions χ_k can be approximated by continuous functions with compact support on the space of lattices X_{d+1}, so we have, as before,

$$P(\mathrm{EST}_{m \times n}(A, c, N) = k) = \eta_N(\chi_K) \xrightarrow[N \to \infty]{} \mu_d(\chi_K).$$

5.1. **Diophantine approximation on curves.** To obtain Erdös-Szüsz-Turán and Kesten distributions for Diophantine approximation on curves, we follow the same procedure above and use the following equidistribution theorem for expanding translates of curves due to N. Shah [38].

Theorem 5.2. *Let $\phi : [a, b] \to \mathbb{R}^n$ be an analytic curve whose image is not contained in a proper affine subspace. Let Γ be a lattice in G. Then for any $x_0 \in G/\Gamma$ and any bounded continuous function f on G/Γ,*

(5.2) $$\lim_{t \to \infty} \frac{1}{b-a} \int_a^b f(g_t u(\phi(s)) x_0) ds = \int_{G/\Gamma} f d\mu.$$

6. Equidistribution on strata

6.1. **Almost everywhere equidistribution.** We prove Theorems 3.1 and 3.2, using Theorem 4.1 and known equidistribution results on the space of lattices. Here, the equivariant assignment is given by

$$S \longmapsto \sum_{z \in \Lambda_S} \delta_z,$$

the counting measure on Λ_S. In the notation of §4, we denote this assignment ν.

We are interested in the Lebesgue measure of the set of $\theta \in [0, 2\pi)$ so that

$$\# \left(r_\theta \Lambda_S \cap H_{A,c,N} \right) = k$$

(or $R_{A,N}$). Applying $g_{\log N}$, and using equivariance, we rewrite this as

$$\# \left(\Lambda_{g_{\log N} r_\theta S} \cap H_{A,c} \right) = k$$

(respectively R_A).

Let $\eta_N(S)$ denote the uniform probability measure $\frac{d\theta}{2\pi}$ on the curve

$$\{g_{\log N} r_\theta S : 0 \le \theta < 2\pi\} \subset \Omega_g,$$

and let $\nu_n = \nu$ denote the equivariant assignment. Thus, we we are in a position to apply Theorem 4.1, with $\mathcal{R} = H_{A,c}$ (or R_A). Theorem 3.1 then follows from Nevo's equidistribution result which states that $\eta_N(S_0) \to \mu_{\mathcal{H}}$ for $\mu_{\mathcal{H}}$-a.e. $S_0 \in \mathcal{H}$.

6.2. **Lattice surfaces.** For Theorem 3.2, we restrict our universe to the subset $\mathrm{SL}(2,\mathbb{R})S_0 \cong \mathrm{SL}(2,\mathbb{R})/\Gamma$, where $\Gamma = \mathrm{SL}(X_0, \omega_0)$. Now, the sequence of measures can be viewed as the measures $\frac{d\theta}{2\pi}$ supported on large circles

$$\{g_{\log N} r_\theta \Gamma : 0 \le \theta < 2\pi\}.$$

By, for example, Dani-Smillie [11], the limiting measure is the Haar probability measure on $\mathrm{SL}(2,\mathbb{R})/\Gamma$, yielding our result. $\qquad\square$

6.3. **Other equivariant assignments.** We note that there are many other equivariant assignments (see [15, §2] and [44]) which can be studied in the context of translation surfaces. Our results, of course, apply to all such assignments.

7. Concluding Remarks

7.1. **Complex Diophantine approximation.** A complex analogue of Dirichlet's theorem states that for every $z \in \mathbb{C} \backslash \mathbb{Q}(i)$, there exist infinitely many $p, q \in \mathbb{Z}[i]$ such that

$$|z - p/q| < 1/|q|^2.$$

One can then consider the modified inequality $|z - p/q| < A/|q|^2$ and the associated Erdös-Szüsz-Turán distribution (the optimal A which allows infinitely many solutions for all z above is $1/\sqrt{3}$, by a result of L. Ford [20]). Complex analogues of our results can be obtained in using analogues of our equidistribution statements. It would be interesting to compute the precise distribution.

7.2. **Distribution of cusp excursions.** More generally, we can consider the distribution of orbits non-uniform lattices $\Lambda \subset \mathrm{SO}(n, 1)$ on the *Clifford plane* Γ_n [4]. In this setting, the Erdös-Szüsz-Turán distribution is closely related to the distribution of cusp excursions for geodesic flow on $\mathbb{H}^n/\Lambda$, a problem we study in [5].

7.3. **S-arithmetic approximation.** Using our methods, one can obtain limiting distributions and moments in both the Erdös-Szüsz-Turán and Kesten settings for p-adic Diophantine approximation, and more generally in the S-arithmetic, characteristic zero setting. In a forthcoming work [7], we show how to obtain such limiting distributions to the positive characteristic setting, proving along the way, equidistribution results of independent interest.

7.4. **Lines.** It is natural to ask if Theorem 2.3, namely the Erdös-Szüsz-Turán and Kesten distributions for nondegenerate curves holds for lines. In this case, the distribution will depend on the Diophantine properties of the slope of the line. In a forthcoming work [5], we comprehensively examine this setting.

7.5. **Multiplicative approximation.** The methods of this paper can be adapted to more general forms of the Erdös-Szüsz-Turán and Kesten distributions which were previously completely inaccessible. For instance, one can consider *multiplicative* and *weighted* forms of Diophantine approximation. In this case, one needs to consider equidistribution of expanding translates of more general diagonal elements. Let $\mathfrak{a}^+$ be the set of k tuples $\mathbf{t} = (t_1, \ldots, t_k) \in \mathbb{R}^k$ such that

$$t_i > 0 \text{ for } 1 \leq i \leq k \text{ and } \sum_{i=1}^{m} t_i = \sum_{j=1}^{n} t_{n+j},$$

and for $\mathbf{t} \in \mathfrak{a}^+$ define

$$(7.1) \qquad\qquad g_{\mathbf{t}} = \mathrm{diag}(e^{t_1}, \ldots, e^{t_m}, e^{-t_{m+1}}, \ldots, e^{-t_k})$$

Applying results of Kleinbock-Margulis [27] (see also [28]) which establish equidistribution of $g_{\mathbf{t}}$ translates on $\mathrm{SL}(d+1, \mathbb{R})/\mathrm{SL}(d+1, \mathbb{Z})$, it should be possible to obtain distributional results in this setting. Similarly, using a result of Shah [39], one can also obtain Erdös-Szüsz-Turán and Kesten distributions in the multiplicative setting for curves.

7.6. **Inhomogeneous approximation and affine lattices.** Our methods can also be applied to the set of *affine* lattices $AX_d = \mathrm{SL}(d, \mathbb{R}) \ltimes \mathbb{R}^d / \mathrm{SL}(d, \mathbb{Z}) \ltimes \mathbb{Z}^d$ to study the distribution of points in affine lattices, with applications to inhomogeneous approximation, see [19] for related results.

7.7. **Cut-and-project quasicrystals.** The fine scale statistics of cut-and-project quasicrystals have been studied numerically by Baake-Götze-Huck-Jacobi [9] and using methods of homogeneous dynamics by Marklof-Strömbergsson [32]. Following the approach of [32], our methods should extend to this setting, and it would be very interesting to work out the explicit distributions.

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